A Lemma on Inequalities

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Abstract

The purpose of this article is to find an upper bound (but not symmetric, as we will see) for the power mean of order k of n numbers using just elementary methods, and to see how to use it in some applications.

Let us start our journey with a known result

Theorem 1. If a and b are real numbers such that $a \geq b \geq 0$ and k is a positive integer, then for all $c_k \in \left(0, \frac{1}{\sqrt[k]{2}-1}\right]$ the following inequality is true:

$$\sqrt[k]{a^k + b^k} \leq a + \frac{b}{c_k}.$$

Proof. We have

$$\sqrt[k]{a^k + b^k} \leq a + \frac{b}{c_k} \Longleftrightarrow a^k + b^k \leq a^k + \sum_{i=1}^{k} \binom{k}{i} \cdot a^{k-i} \cdot \frac{b^i}{c_k^i}$$

From $a \geq b$ it follows that

$$a^k + \sum_{i=1}^{k} \binom{k}{i} \cdot a^{k-i} \cdot \frac{b^i}{c_k^i} \geq a^k + \sum_{i=1}^{k} \binom{k}{i} \cdot \frac{b^k}{c_k^i} = a^k + b^k \cdot \left[\sum_{i=1}^{k} \binom{k}{i} \cdot \left(\frac{1}{c_k}\right)^i\right] = a^k + b^k \cdot \left[\left(1 + \frac{1}{c_k}\right)^k - 1\right] \geq$$

$$\geq a^k + b^k \Longleftrightarrow \left(1 + \frac{1}{c_k}\right)^k - 1 \geq 1 \Longleftrightarrow 1 + \frac{1}{c_k} \geq \sqrt[k]{2} \Longleftrightarrow c_k \leq \frac{1}{\sqrt[k]{2}-1}$$

It is not difficult to see that $c_k = \frac{1}{\sqrt[k]{2}-1}$ is the best constant, because, for $a = b \neq 0$, $a\sqrt[k]{2} \leq a + \dfrac{a}{c_k} \implies \sqrt[k]{2} \leq 1 + \dfrac{1}{c_k} \implies c_k \leq \dfrac{1}{\sqrt[k]{2}-1}$. Equality occurs when $a = b$ or $b = 0$.

Now, let us prove a more general result:

Theorem 2. If $a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n$ are positive real numbers, then the following inequality is satisfied for all $c_k \in \left(0, \frac{1}{\sqrt[k]{2}-1}\right]$:

$$\sqrt[k]{a_0^k + a_1^k + \cdots + a_n^k} \leq a_0 + \frac{a_1}{c_k} + \frac{a_2}{c_k^2} + \cdots + \frac{a_n}{c_k^n}.$$

Equality holds if and only if $a_i = \sqrt[k]{2^{n-i-1}} \cdot m,\ i = 0, 1, \ldots, n-1$, and $m = a_n$.

Proof. Applying **Theorem 1**,

$$\sqrt[k]{a_0^k + a_1^k + \cdots + a_n^k} = \sqrt[k]{a_0 + (a_1^k + \cdots + a_n^k)} \leq a_0 + \frac{\sqrt[k]{a_1^k + \cdots + a_n^k}}{c_k}$$

$$\sqrt[k]{a_1^k + a_2^k + \cdots + a_n^k} = \sqrt[k]{a_1^k + (a_2^k + \cdots + a_n^k)} \leq a_1 + \frac{\sqrt[k]{a_2^k + \cdots + a_n^k}}{c_k}$$

$$\vdots$$

$$\sqrt[k]{a_{n-2}^k + a_{n-1}^k + a_n^k} = \sqrt[k]{a_{n-2}^k + (a_{n-1}^k + a_n^k)} \leq a_{n-2} + \frac{\sqrt[k]{a_{n-1}^k + a_n^k}}{c_k}$$

$$\sqrt[k]{a_{n-1}^k + a_n^k} \leq a_{n-1} + \frac{a_n}{c_k}$$

Combining these n inequalities we get the desired result. Equality occurs if and only if $a_i = \sqrt[k]{a_{i+1}^k + a_{i+2}^k + \cdots + a_n^k},\ i = 0, 1, \ldots, n-1$, i.e. $a_i = \sqrt[k]{2^{n-i-1}} \cdot m,\ i = 0, 1, \ldots, n-1$. So, the main result that we proved in this article is:

$$\sqrt[k]{a_0^k + a_1^k + \cdots + a_n^k} \leq a_0 + a_1(\sqrt[k]{2} - 1) + a_2(\sqrt[k]{2} - 1)^2 + \cdots + a_n(\sqrt[k]{2} - 1)^n,$$

for all $a_0 \geq a_1 \geq \cdots \geq a_n > 0$.

Applications

$\boxed{1}$ Let a and b two non-negative real numbers with $a \geq b$. Prove that the following inequality holds:

$$\sqrt{a^2 + b^2} + \sqrt[3]{a^3 + b^3} + \sqrt[4]{a^4 + b^4} \leq 3a + b.$$

Solution. From **Theorem 1,**

$$\sqrt{a^2 + b^2} \leq a + b(\sqrt{2} - 1)$$
$$\sqrt[3]{a^3 + b^3} \leq a + b(\sqrt[3]{2} - 1)$$
$$\sqrt[4]{a^4 + b^4} \leq a + b(\sqrt[4]{2} - 1)$$

By adding up these inequalities we obtain

$$\sqrt{a^2 + b^2} + \sqrt[3]{a^3 + b^3} + \sqrt[4]{a^4 + b^4} \leq 3a + b(\sqrt{2} + \sqrt[3]{2} + \sqrt[4]{2} - 3) \leq 3a + 0.9 \cdot b \leq 3a + b$$

Equality occurs if and only if $b = 0$. As an observation, this problem cannot be solved by **Power Mean** inequality or **Mildorf's Lemma**. Also, it is not sufficient to observe that $\sqrt[k]{a^k + b^k} \le a + \dfrac{b}{k}$, for $k = 2, 3, 4$, because $\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} \approx 1,083 > 1$.

$\boxed{2}$ Let a, b, c be the side lengths of a triangle, with $a + b + c = 1$, and let $n \ge 2$ be an integer. Prove that

$$\sqrt[n]{a^n + b^n} + \sqrt{b^n + c^n} + \sqrt[n]{c^n + a^n} < 1 + \frac{\sqrt[n]{2}}{2}.$$

[APMO 2003 - Question 4]

Solution. Without loss of generality, suppose that $a \ge b \ge c$. As a, b, c are the side lenghts of a triangle, $b + c > a \Longrightarrow 1 - a > a \Longrightarrow a < \dfrac{1}{2}$ (1)

Now, using **Theorem 1** we obtain $\sqrt[n]{a^n + b^n} \le a + b(\sqrt[n]{2} - 1)$, $\sqrt[n]{b^n + c^n} \le b + c(\sqrt[n]{2} - 1)$, $\sqrt{c^n + a^n} \le a + c(\sqrt[n]{2} - 1)$. Adding up,

$$\sum_{cyc} \sqrt[n]{a^n + b^n} \le 2a + b\sqrt[n]{2} + 2c(\sqrt[n]{2} - 1) = (a + b + c) + a + (b + 2c)(\sqrt[n]{2} - 1) - c$$
$$= 1 + a + (b + 2c)(\sqrt[n]{2} - 1) - c < 1 + a + (b + c)(\sqrt[n]{2} - 1) = 1 + a + (1 - a)(\sqrt[n]{2} - 1)$$
$$= a(2 - \sqrt[n]{2}) + \sqrt[n]{2} < 1 + \frac{\sqrt[n]{2}}{2} \Longleftrightarrow a(2 - \sqrt[n]{2}) < \frac{2 - \sqrt[n]{2}}{2} \Longleftrightarrow a < \frac{1}{2},$$

which is clearly true, from (1).

$\boxed{3}$ Suppose that a, b, c are three non-negative real numbers. Prove that

$$\frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} \ge \frac{10}{(a + b + c)^2}.$$

[Vasile Cîrtoaje, Nguyen Viet Anh]

Solution. Without loss of generality, assume $c = \min(a, b, c)$. From **Theorem 1**, taking into account that $2 \in \left(0, \dfrac{1}{\sqrt{2} - 1}\right]$, we deduce that

$$b^2 + c^2 \le \left(b + \frac{c}{2}\right)^2 = x^2$$
$$a^2 + c^2 \le \left(a + \frac{c}{2}\right)^2 = y^2$$
$$a^2 + b^2 \le \left(a + \frac{c}{2}\right)^2 + \left(b + \frac{c}{2}\right)^2 = x^2 + y^2.$$

Therefore

$$LHS \geq \left(\frac{1}{x^2}+\frac{1}{y^2}\right)\cdot\frac{3}{4}+\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\cdot\frac{1}{4}+\frac{1}{x^2+y^2} \geq$$

$$\geq \frac{\frac{3}{4}\cdot 8}{(x+y)^2}+\frac{1}{2xy}+\frac{1}{x^2+y^2} = \frac{6}{(x+y)^2}+\frac{(x+y)^2}{2xy(x^2+y^2)} \geq \frac{10}{(x+y)^2} \iff$$

$$\iff (x+y)^4 \geq 8xy(x^2+y^2) \iff x^4-4x^3y+6x^2y^2-4xy^3+y^4 \geq 0 \iff$$

$$\iff (x-y)^4 \geq 0.$$

We used **Hölder's** inequality: $(x+y)(x+y)\left(\frac{1}{x^2}+\frac{1}{y^2}\right) \geq 8$. Equality holds for $a=b, c=0$ or permutations.

References

[1] http://www.mathlinks.ro/viewtopic.php?p=446008#446008

[2] Pham Kim Hung, *Secrets in Inequalities*, volume 1, Gil Publishing House, 2007

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